Bidirectional random motion driven by globally coupled noisy active elements in an electric field

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The assembly of the insulating Brownian particles globally coupled due to the macroscopic flow of the liquid with low conductivity has transitions between the states of random motion and random bidirectional and unidirectional motion. The threshold values of the parameters for the transition to random bidirectional motion is found by the effective field method and correspond to those found by Brownian dynamics. The behavior of the assembly is similar to the behavior of different active multistable systems.

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Bistable systems in the presence of thermal noise have gained great interest recently [1–3]. Among the unusual properties of these systems negative stiffness [4], negative friction coefficient [5,6], and resistance due to rectification of Brownian motion [7] should be mentioned. Another interesting feature lies in the symmetry breaking transitions leading to the synchronized behavior of the globally coupled multistable systems. Such transitions have been found for swarming animals [8,9], limit-cycle oscillators [10], chaotic dynamical systems [11], cooperative molecular motors [12], and others. The bidirectional random motion was observed recently in the motility assays for filaments coupled to the assembly of genetically modified molecular motors [13]. This feature of the motion was reproduced by a model of two-state rigidly coupled motors with external noise [14]. Here we consider another system with similar properties. The assembly of insulating Brownian particles in a liquid of low conductivity occupies the space between two solid plates. One of the plates can freely slide with respect to the other in y axis direction. An electric field is applied perpendicularly to the plates in z axis direction (Fig. 1). The direction of the induced dipolar moment of the particle in quiescent fluid due to its negative susceptibility is opposite to the electric field. Since the particle polarization time is finite, then in the vorticity of the flow induced by the motion of the free plate, the dipolar moment acquires the component perpendicular to the field. As a result, the torque appears and creates the rotation of the particle in the direction which sustains the motion of the free plate. This is the so called negative viscosity effect predicted quite a long time ago [15] and confirmed experimentally recently [16]. Due to the continuous energy supply from the batteries, the assembly behaves as an active system with the global coupling between its elements induced by the liquid flow. The theoretical model of the system includes the polarization relaxation equation of the particles [17] and the equation of motion of the system with internal rotations [18]. Since particle rotation in the yz plane can occur in two possible directions, the system possesses the properties of a bistable system. In [19] the case is considered where the bistability of the system arises due to spontaneous rotations of the particles in the electric field above the threshold value. A random thermal motion causing the transitions between the states of a bistable system with the contributions of the opposite sign to the effective viscosity leads to the diminution of the negative viscosity effect. Another situation, as considered here, arises when the rotation of the particles themselves creates the shear flow. In this case due to the global coupling induced by the flow a synchronous rotation of the particles is possible even below the threshold value of the electric field. A constraint imposed on the free plate allows its motion only in the positive or negative y axis direction. Thus a bistable system arises where the bistability is due to the free plate motion. Here we show that the assembly of the noisy active particles is able, in a certain range of the parameters, to create random bidirectional motion of the free plate. The dynamics of the system is characterized by several transitions between random motion, bidirectionality, and directed motion. Transition to the bidirectional random motion may be described by a mean field approximation. In the analysis of the time autocorrelation function of the velocity of the free plate, correlation time in dependence on the physical parameters involved has been found. The increase of the correlation time with the electric field shows the transition to the unidirectional regime when its strength is large enough.

The set of equations includes the stochastic relaxation equation for the dipolar moment of each particle in the external electric field $\vec{E}$ along the z direction [17] ($i=1,\ldots,N$, N is the number of active particles):

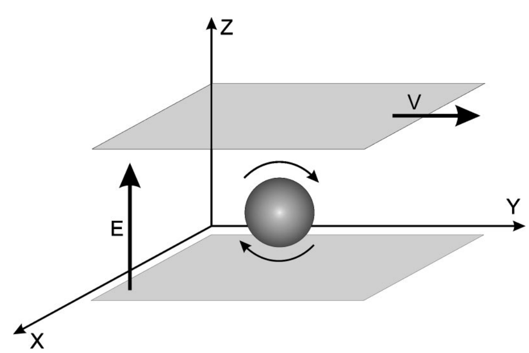

FIG. 1. An insulating particle in a shear flow under the action of an electric field.

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$$\frac{d\vec{P}_i}{dt} = [\vec{\Omega}_i \times \vec{P}_i] - \frac{1}{\tau}(\vec{P}_i - (\chi_0 - \chi_\infty)\vec{E}), \quad (1)$$

$$\vec{\Omega}_i = \vec{\Omega}_0 + \frac{1}{\alpha_r}[\vec{P}_i \times \vec{E}] + \vec{\Omega}_{ri}. \quad (2)$$

Here $\chi_{0,\infty}$ are the susceptibilities of particle polarization at low and high electric field frequencies correspondingly:

$$\chi_0 = \frac{3V\epsilon_1}{4\pi}\frac{\gamma_2 - \gamma_1}{\gamma_2 + 2\gamma_1},$$

$$\chi_\infty = \frac{3V\epsilon_1}{4\pi}\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + 2\epsilon_1},$$

$$\tau = \epsilon_2 + 2\epsilon_1/4\pi(\gamma_2 + 2\gamma_1),$$

is the Maxwell relaxation time, α_r is the rotational friction coefficient of the particle, $\vec{\Omega}_0$ is the vorticity of flow, V is the volume of the particle, index 1 refers to the fluid and index 2 to the particle. The vorticity of the macroscopic flow $\vec{\Omega}_0$ in assumption of the Couetta flow between two plates is expressed through the velocity of the free plate v as $\vec{\Omega}_0 = -(v/2h)\vec{e}_x$ (h is the distance between the plates). The velocity of the free plate is determined by the balance of the viscous force and the force due to N active polarizable particles. Since the torque on the particle is $[\vec{P}_i \times \vec{E}]_x$, then the force applied to the liquid and as consequence to the plate in y axis direction can be written as $1/2d[\vec{P}_i \times \vec{E}]_x$, where d is the characteristic length with the order of magnitude of the particle diameter. Thus the force balance on the free plate reads

$$-\alpha v - \sum_{i=1}^{N}\frac{1}{2d}[\vec{P}_i \times \vec{E}]_x = 0. \quad (3)$$

Here $\alpha = \eta S/h$, S is the area of the plate, η is the viscosity of the liquid.

Introducing as a characteristic time scale the Maxwell relaxation time τ, $\vec{P} = -(\chi_0 - \chi_\infty)E\tilde{\vec{P}}$ and introducing the unit vector $\vec{n}$ along the polarization direction $\tilde{\vec{P}} = \tilde{P}\vec{n}$ (tildas further are omitted) the set of dimensionless equations for the globally coupled noisy active particles is $(i = 1, \ldots, N)$

$$\frac{d\vec{n}_i}{dt} = [\vec{\Omega}_{ni} \times \vec{n}_i], \quad (4)$$

$$\frac{dP_i}{dt} = -(P_i + \vec{e}\cdot\vec{n}_i), \quad (5)$$

$$\vec{\Omega}_{ni} = a\frac{E^2}{E_c^2}\frac{1}{N}\sum_{j=1}^{N} P_{jy}\vec{e}_x + \frac{1}{P_i}\left(1 - \frac{E^2}{E_c^2}P_i^2\right)[\vec{e} \times \vec{n}_i] + \vec{\Omega}_{ri}. \quad (6)$$

Here the critical electric field $E_c^2 = -[\alpha_r/\tau(\chi_0 - \chi_\infty)]$ is introduced. At $E > E_c$ spontaneous rotations of the single particles, the so-called von Quinke effect [19] arises. The parameter $a = \alpha_r N/4hd\alpha$ characterizes the global coupling of the elements. In the continuum limit, the parameter a reduces to $n\alpha_r/4\eta$, where n is the number of particles per volume unit [15]. Noise $\vec{\Omega}_{ri}$ for the different particles is uncorrelated.

Equation $\vec{\Omega}_{ni} = 0$ in the mean field approximation $1/N\Sigma_{j=1}^{N} P_{jy} = P_y$ and the absence of the noise for P_i gives $P_i^2 = [1/(1+a)](E^2/E_c^2)$. Thus the solution $|P_i| < 1$ is possible at the electric field above the critical value $E_* = E_c/\sqrt{1+a}$, which at $a > 0$ is less than the threshold value of von Quinke effect E_c. The velocity of the free plate scaled with $2h/\tau$ inducing the synchronous rotations of the particles at $E > E_*$ found from Eq. (3) is

$$v = \pm \frac{a}{1+a}\sqrt{\frac{E^2}{E_*^2} - 1}.$$

Thus the motion of the free plate occurs in positive or negative y axis direction at $E > E_*$. In the presence of the thermal noise transitions between these two states are possible. The regimes arising in this case can be found by the numerical solution of the set of stochastic differential equations (4)–(6). It is carried out by the Brownian dynamics method as described in [19].

Dispersions of the random angles of the particle rotation according to the fluctuation-dissipation theorem are given by ($k = 1, 2$ denotes the components of the random angular velocity $\vec{\Omega}_{ri}$ perpendicular to instantaneous direction of $\vec{n}$)

$$\langle(\Omega_{ri}^{(k)}\Delta t)^2\rangle = 2\frac{\tau}{\tau_B}\Delta t.$$

Here $\tau_B = \alpha_r/k_B T$ is the characteristic Brownian time. Since in our case τ_B/τ is quite large then adiabatic approximation [20] according to which $P_i = -\vec{e}\cdot\vec{n}_i$ may be applied. In this approximation the Fokker-Planck equation for the probability distribution function $W(\vec{n})$ corresponding to (5) and (6) in the case when global coupling is absent and the particles are independent is ($\vec{K}_n = [\vec{n} \times (\partial/\partial\vec{n})]$)

$$\frac{\partial W}{\partial t} = \frac{\tau}{\alpha_r}\vec{K}_n(\vec{K}_n E W) + \frac{\tau}{\tau_B}\vec{K}_n^2 W. \quad (7)$$

Here the potential E of the active element reads

$$E = \frac{\alpha_r}{\tau}\left(-\ln|\vec{e}\vec{n}| + \frac{1}{2}\frac{E^2}{E_c^2}(\vec{e}\cdot\vec{n})^2\right). \quad (8)$$

Equation (7) as a steady solution has

$$W_0(n_z) = Q^{-1}|\vec{e}\vec{n}|^{\tau_B/\tau}\exp\left(-\frac{1}{2}\frac{\tau_B}{\tau}\frac{E^2}{E_c^2}(\vec{e}\cdot\vec{n})^2\right). \quad (9)$$

Normalized distribution $W_z = W_0(n_z)/W_0(1)$ of n_z values obtained according to the Brownian dynamics and averaged for 10 realizations of 5×10^6 long runs with time step $\Delta t = 0.01$ for $b = (\tau_B/\tau)(E^2/E_c^2) = 0.4$ and calculated according to the relation (9) is shown in Fig. 2. The distribution W_y of n_y values is calculated according to the relation

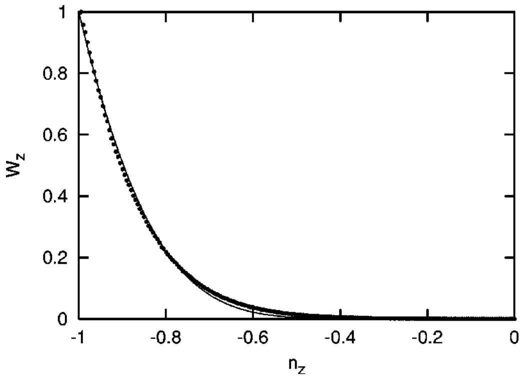

FIG. 2. Distribution of the n_z component. $N=1$, $a=0$, $\tau_B/\tau=10$. Averaged by 10 realizations of 5×10^6 time step runs with $\Delta t=0.01$. $E^2/E_c^2=0.4$.

$$W_{0y}(n_y) = Q^{-1}\int \delta(n_y - \sin\vartheta\sin\varphi)|\cos\vartheta|^{\tau_B/\tau} \times\exp(-b\sin^2\vartheta)\sin\vartheta d\vartheta d\varphi. \qquad (10)$$

The normalized probability distribution $W_y = W_{0y}(n_y)/W_{0y}(0)$ found by Brownian dynamics and averaged for 10 realizations of 5×10^6 long runs with time step $\Delta t=0.01$ for $b=(\tau_B/\tau)(E^2/E_c^2)=0.4$ and calculated according to the relation (10) is shown in Fig. 3. We see that in the absence of the global coupling the distribution of the transversal to the electric field component of the dipolar moment of the particles is unimodal.

At nonzero values of the coupling parameter a a transition to the bimodal distribution for the transversal component of polarization takes place. In the mean field approximation $1/N\sum_{i=1}^{N}P_{iy}=P_y$ valid if the number of particles in assembly is large enough, the Fokker-Planck equation reads

$$\frac{\partial W}{\partial t} = -\vec{K}_n(\vec{\Omega}_0 W) + \frac{\tau}{\alpha_r}\vec{K}(\vec{K}_n E W) + \frac{\tau}{\tau_B}\vec{K}_n^2 W. \qquad (11)$$

Here $\vec{\Omega}_0 = a(E^2/E_c^2)P([\vec{n}\times\vec{e}]\cdot\vec{e}_x)\vec{e}_x$, where $P=-n_z$ according to the adiabatic approximation. The critical parameters for the transition to the bimodal distribution can be found in the

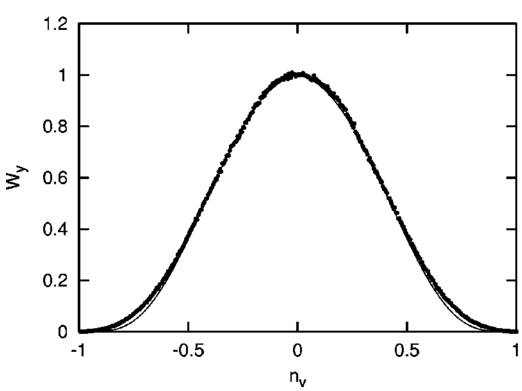

FIG. 3. Distribution of the n_y component. $N=1$, $a=0$, $\tau_B/\tau=10$. Averaged by 10 realizations of 5×10^6 time step runs with $\Delta t=0.01$. $E^2/E_c^2=0.4$.

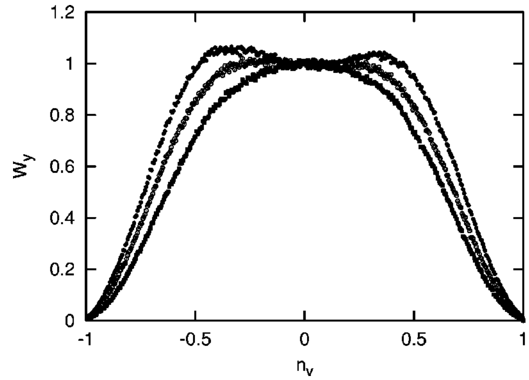

FIG. 4. Transition to the bimodal distribution. $N=40$, $a=3$, $\tau_B/\tau=10$. Averaged by 10 realizations of 5×10^6 time step runs with $\Delta t=0.01$. $E^2/E_c^2=0.43$ (squares); $E^2/E_c^2=0.44$ (empty circles); $E^2/E_c^2=0.45$ (filled circles).

effective field approximation which has been successfully applied to the problems of the magnetic relaxation in magnetic colloids even in situations far from the thermodynamic equilibrium [21,22]. According to it

$$W = Q^{-1}\exp\left(-\frac{E}{k_B T} + \frac{\vec{\lambda}\cdot\vec{n}}{k_B T}\right),$$

where the value of the effective field $\vec{\lambda}$ is fixed by the condition

$$\int \vec{n}W(\vec{n})d^2\vec{n} = \langle\vec{n}\rangle.$$

The equation for the effective field parameter $\vec{\lambda}$ is found by multiplying Eq. (11) with $\vec{n}$ and taking the average. Accounting for the antihermitian property of operator $\vec{K}_n$ we obtain ($\langle\rangle$ denotes the average with respect to the distribution function W)

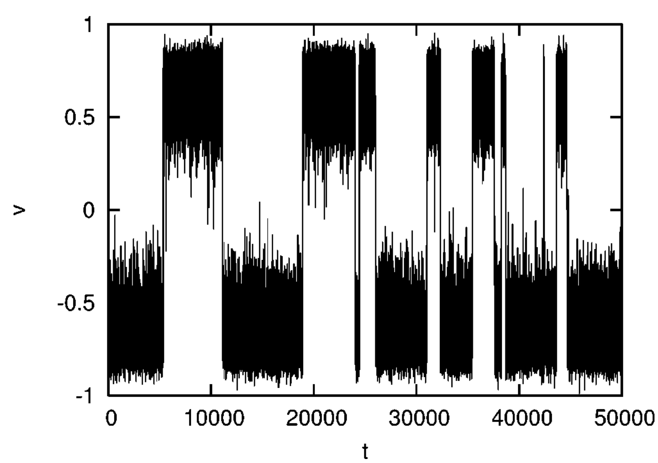

FIG. 5. Random bidirectional motion of free plate. $N=20$, $a=3$, 5×10^6 time steps with $\Delta t=0.01$ are shown. $E^2/E_c^2=0.54$, $\tau_B/\tau=10$.

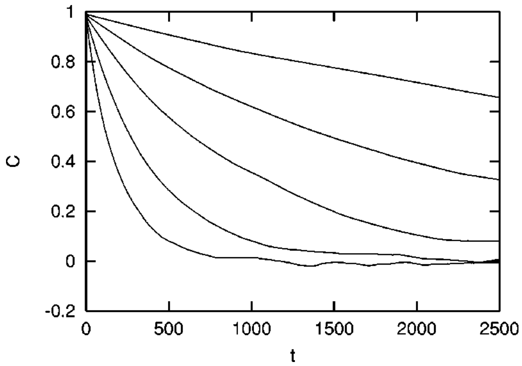

FIG. 6. Plate velocity autocorrelation function. $N=20$, $a=3$, $\tau_B/\tau=20$. Averaged by 10 realizations of 5×10^6 time steps with $\Delta t=0.01$. E^2/E_c^2 changes from 0.4 to 0.36 starting from above.

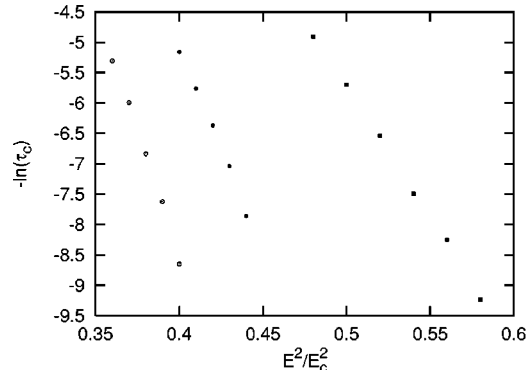

FIG. 7. Plate velocity correlation time in dependence on E^2/E_c^2 for $\tau_B/\tau=20$ (open circles); $\tau_B/\tau=15$ (filled circles); $\tau_B/\tau=10$ (squares). $N=20$, $a=3$. Averaged by 10 realizations of 5×10^6 time step runs with $\Delta t=0.01$.

$$\frac{\partial\langle\vec{n}\rangle}{\partial t}=\langle[\vec{\Omega}_0\times\vec{n}]\rangle-\frac{\tau}{\alpha_r}(\vec{\lambda}-\langle\vec{n}(\vec{n}\cdot\vec{\lambda})\rangle). \quad (12)$$

This is a nonlinear equation for the effective field $\vec{\lambda}$. In the linear approximation, Eq. (12) allows one to study the stability of the unimodal distribution with respect to the formation of the state with transversal mean polarization. In this case

$$W=W_0\left(1+\frac{\lambda_y n_y}{k_BT}\right)$$

and Eq. (12) gives

$$\frac{\partial\langle n_y\rangle}{\partial t}=\frac{\tau}{\alpha_r}\lambda_y\left(a\frac{\tau_B}{\tau}\frac{E^2}{E_c^2}\langle n_z^2n_y^2\rangle_0-(1-\langle n_z^2\rangle_0)\right). \quad (13)$$

Here $\langle\rangle_0$ denotes the moments of the distribution function W_0. The condition of the instability of the unimodal distribution reads

$$a\frac{\tau_B}{\tau}\frac{E^2}{E_c^2}\langle n_z^2n_y^2\rangle_0-(1-\langle n_z^2\rangle_0)>0.$$

The threshold value of the parameter $b=(\tau_B/\tau)(E^2/E_c^2)$ for the transition to the bimodal distribution is found from the solution of the equation

$$ab=\frac{f_0(\tau_B/\tau,b)+f_1(\tau_B/\tau,b)}{f_1(\tau_B/\tau,b)-f_2(\tau_B/\tau,b)}. \quad (14)$$

Here $f_n(\tau_B/\tau,b)=\int_0^1 x^{\tau_B/\tau+2n}\exp\left(-\frac{1}{2}bx^2\right)dx$. The threshold value of the parameter b found by the solution of Eq. (14) coincides reasonably well with that found by Brownian dynamics simulation. The distributions of n_y found numerically in the case $N=40$ and $a=3;\tau_B/\tau=10$ for three values of E^2/E_c^2 are shown in Fig. 4. We see that the transition to the bimodal distribution occurs for E^2/E_c^2 between 0.44 and 0.45. This value coincides reasonably well with that found from the solution of Eq. (14), $(E^2/E_c^2)_*=0.4432$. If the number of particles N is not large enough, the threshold value $(E^2/E_c^2)_*$ diminishes with respect to that found in mean field approximation. Above the critical value $(E^2/E_c^2)_*$ beautiful random bidirectional motion of the free plate arises as shown in Fig. 5.

The characteristic time of bidirectional oscillations may be found from the study of the time autocorrelation function of the free plate velocity $C(t)=\langle v(0)v(t)\rangle/\langle v(0)^2\rangle$. For particular values of the parameters they are shown in Fig. 6. For the time t not too large $C(t)$ obeys the exponential law $\exp(-t/\tau_c)$. The characteristic plate velocity correlation time τ_c is found from semilogarithmic plot of autocorrelation functions. Its dependence on E^2/E_c^2 for several values of τ_B/τ in semilogarithmic coordinates is given in Fig. 7. These results show that at the increase of the parameter E^2/E_c^2, the velocity relaxation time increases drastically. Due to this the dynamics of the system becomes unidirectional if E^2/E_c^2 is large enough, for example at $E^2/E_c^2=0.65$; $N=20;\tau_B/\tau=10;a=3$ unidirectionality is conserved for 5×10^6 time steps with $\Delta t=0.01$ long run.

Thus, the results obtained show that the assembly of the insulating particles in liquid with low conductivity allows one to create globally coupled system of noisy active elements. Its properties are similar to those found in different active multistable systems. It allows one to study the transition rates between the steady states of the non-equilibrium system which have attracted great interest.

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